STRONG CLEANNESS OF MATRIX RINGS OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative local ring. It is proved that R is Henselian if and only if each R-algebra which is a direct limit of module finite R-algebras is strongly clean. So, the matrix ring $\mathbb{M}_n(R)$ is strongly clean for each integer $n > 0$ if R is Henselian and we show that the converse holds if either the residue class field of R is algebraically closed or R is an integrally closed domain or R is a valuation ring. It is also shown that each R-algebra which is locally a direct limit of module-finite algebras, is strongly clean if R is a π-regular commutative ring.

As in [10] a ring R is called **clean** if each element of R is the sum of an idempotent and a unit. In [8] Han and Nicholson proved that a ring R is clean if and only if $\mathbb{M}_n(R)$ is clean for every integer $n \geq 1$. It is easy to check that each local ring is clean and consequently every matrix ring over a local ring is clean. On the other hand a ring R is called **strongly clean** if each element of R is the sum of an idempotent and a unit that commute. Recently, in [12], Chen and Wang gave an example of a commutative local ring R with $\mathbb{M}_2(R)$ not strongly clean. This motivates the following interesting question: what are the commutative local rings R for which $\mathbb{M}_n(R)$ is strongly clean for each integer $n \geq 1$? In [4], Chen, Yang and Zhou gave a complete characterization of commutative local rings R with $\mathbb{M}_2(R)$ strongly clean. So, from their results and their examples, it is reasonable to conjecture that the Henselian rings are the only commutative local rings R with $\mathbb{M}_n(R)$ strongly clean for each integer $n \geq 1$. In this note we give a partial answer to this problem. If R is Henselian then $\mathbb{M}_n(R)$ is strongly clean for each integer $n \geq 1$ and the converse holds if R is an integrally closed domain, a valuation ring or if its residue class field is algebraically closed.

All rings in this paper are associative with unity. By [11, Chapitre I] a commutative local ring R is said to be **Henselian** if each commutative module-finite R-algebra is a finite product of local rings. It was G. Azumaya ([1]) who first studied this property which was then developed by M. Nagata ([9]). The following theorem gives a new characterization of Henselian rings.

Theorem 1. *Let R be a commutative local ring. Then the following conditions are equivalent:*

1. *R is Henselian;*
2. *For each R-algebra A which is a direct limit of module-finite algebras and for each integer $n \geq 1$, the matrix ring $\mathbb{M}_n(A)$ is strongly clean;*
3. *Each R-algebra A which is a direct limit of module-finite algebras is clean.*

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Proof. (1) $\Rightarrow$ (2). Let A be a direct limit of module-finite R-algebras and $a \in \mathbb{M}_n(A)$. Then $R[a]$ is a commutative module-finite R-algebra. Since R is Henselian, $R[a]$ is a finite direct product of local rings. So $R[a]$ is clean. Hence a is a sum of an idempotent and a unit that commute.

It is obvious that (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let A be a commutative module-finite R-algebra and let $J(A)$ be its Jacobson radical. Since $J(R)A \subseteq J(A)$, where $J(R)$ is the Jacobson radical of R, we deduce that $A/J(A)$ is semisimple artinian. By [10, Propositions 1.8 and 1.5] idempotents can be lifted modulo $J(A)$. Hence A is semi-perfect. It follows that A is a finite product of local rings, whence R is Henselian. □

Let $\mathcal{P}$ be a ring property. We say that an algebra A over a commutative ring R is **locally** $\mathcal{P}$ if A_P satisfies $\mathcal{P}$ for each maximal ideal P of R.

Corollary 2. *Let R be a commutative ring. Then the following conditions are equivalent:*

1. *R is clean and locally Henselian;*
2. *For each R-algebra A which is locally a direct limit of module-finite algebras and for each integer $n \geq 1$, $\mathbb{M}_n(A)$ is strongly clean;*
3. *Each R-algebra A which is locally a direct limit of module-finite algebras is clean.*

Proof. (1) $\Rightarrow$ (2). Let A be an R-algebra which is locally a direct limit of module-finite algebras and $a \in \mathbb{M}_n(A)$. Consider the following polynomial equations: $E + U = a,\ E^2 = E,\ UV = 1,\ VU = 1,\ EU = UE$. By Theorem 1 these equations have a solution in $\mathbb{M}_n(A_P)$, for each maximal ideal P of R. So, by [5, Theorem I.1] they have a solution in $\mathbb{M}_n(A)$ too.

It is obvious that (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let P be a maximal ideal of R and let A be a module-finite R_P-algebra. Since R is clean, the natural map $R \to R_P$ is surjective by [5, Theorem I.1 and Proposition III.1]. So A is a module-finite R-algebra. It follows that A is clean. By Theorem 1 R_P is Henselian. □

A ring R is said to be **strongly π-regular** if, for each $r \in R$, there exist $s \in R$ and an integer $q \geq 1$ such that $r^q = r^{q+1}s$.

Corollary 3. *Let R be a strongly π-regular commutative ring. Then, for each R-algebra A which is locally a direct limit of module-finite algebras and for each integer $n \geq 1$, the matrix ring $\mathbb{M}_n(A)$ is strongly clean.*

Proof. It is known that R is clean and that each prime ideal is maximal. So, for every maximal P, PR_P is a nilideal of R_P. Hence R_P is Henselian. We conclude by Corollary 2. □

By [6, Théorème 1] each strongly π-regular R satisfies the following condition: for each $r \in R$, there exist $s \in R$ and an integer $q \geq 1$ such that $r^q = sr^{q+1}$. Moreover, by [3, Proposition 2.6.iii)] each strongly π-regular ring is strongly clean. So, Corollary 3 is also a consequence of the following proposition. (Probably, this proposition is already known).

Proposition 4. *Let R be a strongly π-regular commutative ring. Then, for each R-algebra A which is locally a direct limit of module-finite algebras and for each integer $n \geq 1$, the matrix ring $\mathbb{M}_n(A)$ is strongly π-regular.*

Proof. Let $S = \mathbb{M}_n(A)$ and $s \in S$. Then $R[s]$ is locally a module-finite algebra. It is easy to prove that each prime ideal of $R[s]$ is maximal. Consequently $R[s]$ is strongly π-regular. So, S is strongly π-regular too. □

The following lemma will be useful in the sequel.

Lemma 5. *Let R be a commutative local ring with maximal ideal P. Let n be an integer > 1 such that $\mathbb{M}_n(R)$ is strongly clean. Let f be a monic polynomial of degree n with coefficients in R such that $f(0) \in P$ and $f(a) \in P$ for some $a \in R \setminus P$. Then f is reducible.*

Proof. Let $A \in \mathbb{M}_n(R)$ such that its characteristic polynomial is f, i.e. $f = \det(XI_n - A)$, where I_n is the unit element of $\mathbb{M}_n(R)$. Then $A = E + U$ where E is idempotent, U is invertible and $EU = UE$. First we assume that $a = 1$. So, 0 and 1 are eigenvalues of $\overline{A}$ the reduction of A modulo P. Consequently A and $A - I_n$ are not invertible. It follows that $E \neq I_n$ and $E \neq 0_{n,n}$ where $0_{p,q}$ is the $p \times q$ matrix whose coefficients are 0. Let F be a free R-module of rank n and let ϵ be the endomorphism of F for which E is the matrix associated with respect to some basis. Then $F = \mathrm{Im}\ \epsilon \oplus \mathrm{Ker}\ \epsilon$. Moreover $\mathrm{Im}\ \epsilon$ and $\mathrm{Ker}\ \epsilon$ are free because R is local. Consequently there exists a $n \times n$ invertible matrix Q such that:

$$QEQ^{-1} = B = \begin{pmatrix} I_p & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{pmatrix}$$

where p is an integer such that $0 < p < n$ and $q = n - p$. Since E and A commute, then B and QAQ^{-1} commute too. So, QAQ^{-1} is of the form:

$$QAQ^{-1} = \begin{pmatrix} C & 0_{p,q} \\ 0_{q,p} & D \end{pmatrix}$$

where C is a $p \times p$ matrix and D is a $q \times q$ matrix. We deduce that f is the product of the characteristic polynomial g of C with the characteristic polynomial h of D. Let us observe that $(C - I_p)$ and D are invertible. So, $g(1) \notin P$, $h(0) \notin P$, $g(0) \in P$ and $h(1) \in P$. Now suppose that $a \neq 1$. Then $a^{-n}f(X) = g(Y)$ where $Y = a^{-1}X$ and g is a monic polynomial of degree n. We easily check that $g(1) \in P$ and $g(0) \in P$. It follows that g is reducible, whence f is reducible too. □

A commutative ring R is a **valuation ring** (respectively **arithmetic**) if its lattice of ideals is totally ordered by inclusion (respectively distributive).

Theorem 6. *Let R be a local commutative ring with maximal ideal P and with residue class field k. Consider the following two conditions:*

(1) *R is Henselian;*
(2) *the matrix ring $\mathbb{M}_n(R)$ is strongly clean $\forall n \in \mathbb{N}^*$.*

Then (1) $\Rightarrow$ (2) and the converse holds if R satisfies one of the following properties:

(a) *k is algebraically closed;*
(b) *R is an integrally closed domain;*
(c) *R is a valuation ring.*

Proof. By Theorem 1 it remains to prove that (2) implies (1) when one of (a), (b) or (c) is valid. We will use [2, Theorem 1.4] and [7, Theorem II.7.3.(iv)]. Consider the polynomial $f = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$ and assume that $\exists m,\ 1 \leq m < n$ such that $c_m \notin P$ and $c_i \in P,\ \forall i < m$. Since $c_0 \in P$, we see that $f(0) \in P$.

Hence, if k is algebraically closed, $\exists a \in R \setminus P$ such that $f(a) \in P$. By Lemma 5 f is reducible. So, by [2, Theorem 1.4] R is Henselian.

If R is an integrally closed domain, we take $m = n - 1$ for proving the condition (iv) of [7, Theorem II.7.3]. In this case $f(-c_{n-1}) \in P$. By Lemma 5 (possibly applied several times) f satisfies the condition (iv) of [7, Theorem II.7.3]. Hence R is Henselian.

Assume that R is a valuation ring. Let N be the nilradical of R and let $R' = R/N$. We know that R is Henselian if and only if R' is Henselian too. For each $n \in \mathbb{N}^*$, $\mathbb{M}_n(R')$ is strongly clean. Since R' is a valuation domain, R' is integrally closed. It follows that R' and R are Henselian. □

Corollary 7. *Let R be an arithmetic commutative ring. Then the following conditions are equivalent:*

1. *R is clean and locally Henselian;*
2. *the matrix ring $\mathbb{M}_n(R)$ is strongly clean $\forall n \in \mathbb{N}^*$.*

Proof. By Corollary 2 it remains to show (2) $\Rightarrow$ (1). Let P be a maximal ideal of R. Since R is clean the natural map $R \to R_P$ is surjective by [5, Theorem I.1 and Proposition III.1]. So, $\mathbb{M}_n(R_P)$ is strongly clean $\forall n \in \mathbb{N}^*$. Theorem 6 can be applied because R_P is a valuation ring. We conclude that R_P is Henselian. □

The following generalization of [4, Theorem 8] holds even if the properties $(a), (b), (c)$ of Theorem 6 are not satisfied.

Theorem 8. *Let R be a local commutative ring with maximal ideal P and with residue class field k. Let p be an integer such that $2 \leq p \leq 5$. Then the following conditions are equivalent:*

1. *$\mathbb{M}_n(R)$ is strongly clean $\forall n,\ 2 \leq n \leq p$;*
2. *each monic polynomial f of degree $n,\ 2 \leq n \leq p$, for which $f(0) \in P$ and $f(1) \in P$, is reducible.*

Proof. By Lemma 5 it remains to prove that (2) $\Rightarrow$ (1). Let $A \in \mathbb{M}_n(R)$. We denote by f the characteristic polynomial of A. If A is invertible then $A = 0_{n,n} + A$. If $A - I_n$ is invertible then $A = I_n + (A - I_n)$. So, we may assume that A and $(A - I_n)$ are not invertible. It follows that $f(0) \in P$ and $f(1) \in P$. Then, $f = gh$ where g and h are monic polynomials of degree ≥ 1. We may assume that $g(0) \in P,\ g(1) \notin P,\ h(0) \notin P$ and $h(1) \in P$ (possibly by applying condition (2) several times). We denote by $\bar{f},\ \bar{g},\ \bar{h}$ the images of $f,\ g,\ h$ by the natural map $R[X] \to k[X]$. If $\bar{g}$ and $\bar{h}$ have a common factor of degree ≥ 1 then this factor is of degree 1 because $n \leq 5$. In this case $\exists a \in R \setminus P$ such that $g(a) \in P$ and $h(a) \in P$. As in the proof of Lemma 5 we show that g is reducible. Hence, after changing g and h, we get that $\bar{g}$ and $\bar{h}$ have no common divisor of degree ≥ 1. It follows that there exist two polynomials u and v with coefficients in R such that $\bar{u}\bar{g} + \bar{v}\bar{h} = 1$. Since $PR[A]$ is contained in the Jacobson radical of $R[A]$, we may assume that $u(A)g(A) + v(A)h(A) = I_n$. We put $e = vh$. Then we easily check that $e(A)$ is idempotent. It remains to show that $(A - e(A))$ is invertible. It is enough to prove that $(\bar{A} - \bar{e}(\bar{A}))$ is invertible because $P\mathbb{M}_n(R)$ is the Jacobson radical of $\mathbb{M}_n(R)$. Let V be a vector space of dimension n over k and let $\mathcal{B}$ be a basis of V. Let α be the endomorphism of V for which $\bar{A}$ is the matrix associated with respect to $\mathcal{B}$. We put $\epsilon = \bar{e}(\alpha)$. Since V has finite dimension, it is sufficient to show that $(\alpha - \epsilon)$ is injective. Let $w \in V$ such that $\alpha(w) = \epsilon(w)$. It follows that

$\alpha(\epsilon(w)) = \epsilon(\alpha(w)) = \epsilon^2(w) = \epsilon(w)$. Since $\bar{e}$ is divisible by $(X - \bar{1})$ we get that $\epsilon(w) = 0$. So, $\alpha(w) = 0$. We deduce that $\epsilon(w) = w$ because $\bar{e} - \bar{1}$ is divisible by X. Hence $w = 0$. $\square$

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